AN EXPANDED LUTTINGER MODEL

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2/15/2012

Abstract:

This paper generalizes Luttinger's model by introducing curvature ($\frac{d^2\varepsilon(k)}{dk^2} \neq 0$) into the kinetic energy. An exact solution for arbitrary interactions is still possible in principle, but it now requires disentangling the eigenvalue spectrum of an *harmonic string* of interacting boson fields at *each* value of q. The additional boson fields, extracted from the excitation spectrum of the Fermi sea, are self-selected according to the nature and strength of the dispersion.

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INTRODUCTION. After it was first solved exactly by "bosonization" [1] and its solution inverted later by "fermionization," [2] Luttinger's model[3] of interacting fermions continued to deliver a huge amount of information concerning two-body correlations and other properties – far more than one might have expected from the standard apparatus of mathematical physics.[4] No wonder! In this reductive two-branch model of SU(2) fermions in 1D, half move to the right at a constant velocity while the others move left, also at constant speed. A right-going fermion in a plane wave state k has kinetic energy $\varepsilon(k) = c(k - k_F)$, and a left-going fermion has kinetic energy $-c(k + k_F)$ over a common range $-\infty < k < +\infty$. For reasons related to the constant speed and to the separate number-conservation of the right- and of the left-going particles, even in the presence of arbitrary two-body forces $U(x - x')$ the Hamiltonian of this model is easily diagonalized.

The present work expands both the model and its solution to the many instances in which the speed of the fermions is *not* constant, barring only "backscatterings" from one branch to the other.[5] We show that under these circumstances it is still possible to transform the interacting fermions into a multitude of bosons – all culled out of a common Fermi sea. Once the dispersion $\frac{d^2\varepsilon(k)}{dk^2}$ ceases to be identically zero, it becomes necessary to consider not just a single harmonic oscillator at each value of the momentum transfer q but a full-fledged harmonic *string* at each q. Quadratic forms in bosons (or, for that matter, fermions) can *always* be diagonalized exactly and all their eigenstates determined although not always *analytically* in closed form. The extended model presented below remains - in principle – a quadratic form, hence exactly solvable. In examples such as the truncated 2-site string version explicitly worked out below, *Mathematica* on a home *PC* proved sufficiently powerful to extract the roots in closed form and to plot them.

REVIEW OF THE (ORIGINAL) DISPERSIONLESS MODEL. The motional (kinetic) energy of individual fermions in the Luttinger model is given by $\varepsilon_\tau(k)=c(\tau k-k_F)$, where $\tau=\pm1$ labels the right (+)and left (–) going particles. The relevant operator is,

$$KE=\sum_k\sum_\tau\sum_\sigma \varepsilon_\tau(k)c^\dagger_{\sigma,\tau}(k)c_{\sigma,\tau}(k) \quad , \quad \text{where } \varepsilon_\tau(k)=c(\tau k-k_F) \tag{1}$$

where spin index $\sigma=\uparrow or \downarrow$ (or $\pm\frac{1}{2}$) .

The interaction Hamiltonian H_2 involves $U(x\text{-}x')$, the potential that connects fermions at x and x', *via* its Fourier transform $V(q)/L$. Delta-function interactions, $V(q)$ = constant, are distinguished from longer-ranged interactions $V(q) \propto 1/|q|$ or $1/|q|^2$ by powers of $1/|q|$. The density operator for the right-hand-goers (τ=+1) can be written in terms of density creation/annihilation bosonic-type operators, as follows:

$$\rho_+(q)=\sum_{k,\sigma}c^+_{\sigma,+}(k-q/2)c_{\sigma,+}(k+q/2)=\begin{cases}\sqrt{\dfrac{qL}{2\pi}}\displaystyle\sum_\sigma a_\sigma(q) & \text{if } q>0\\ \sqrt{\dfrac{-qL}{2\pi}}\displaystyle\sum_\sigma a^\dagger_\sigma(-q) & \text{if } q<0\end{cases}$$

with complementary relations for the left-goers. The commutator bracket relations satisfied by the a's are the obvious ones:

$[a_\sigma(q),a_{\sigma'}(q')]=0=[a^\dagger_{\sigma'}(q'),a^\dagger_\sigma(q)]$ and $[a_\sigma(q),a^\dagger_{\sigma'}(q')]=\delta_{\sigma,\sigma'}\delta_{q,q'}$.

When rewritten in such operators, the interaction Hamiltonian is:

$$H_2=\frac{1}{2}\times\frac{1}{2\pi}\sum_{q=-\infty}^{+\infty}\sum_\sigma\sum_{\sigma'}V(q)|q|\Big(\big(a^\dagger_\sigma(q)+a_{-\sigma}(-q)\big)a_{\sigma'}(q)+H.c.\Big) \quad . \tag{2}$$

NEGLECT OF EXCHANGE TERMS. In order to remain simple and solvable, the above formulation omits *exchange* interactions. To date, most papers dealing with the Luttinger model have ignored the effects both of exchange terms in the interactions and of dispersion in the kinetic energy. In this paper we study the effects of dispersion, but the reader may ask, are not exchange terms equally vital to our understanding? In the case of delta function interactions, the exchange correction completely cancels all self-interactions $\sum_{\sigma} a_{\sigma}^{\dagger}(q)a_{\sigma}(q)$ in Eq. (2). In the absence of dispersion this *could* be understood as a renormalization of the motional energy. What remains is a variant of the sigma model. However, for arbitrary two-body potentials ($V(q) \neq$ constant), the dynamics of exchange corrections can rapidly become very ugly.

For this reason we have chosen to ignore exchange in the remainder of this work, although we shall return to the topic elsewhere.

IMPORTANT FEATURES OF LUTTINGER MODEL. The total Hamiltonian $H = KE+ H_2$ decouples into non-overlapping sectors labeled q. Each sector involves only 4 boson operators: $a_{\sigma}(q), a_{-\sigma}^{\dagger}(-q), a_{-\sigma}(-q), a_{\sigma}^{\dagger}(q)$. Assuming $cq > 0$, the commutator bracket equations of motion $[a_{\sigma}(q), KE] = cq\, a_{\sigma}(q)$ and $[a_{\sigma}^{\dagger}(q), KE] = -cq a_{\sigma}^{\dagger}(q)$ show these to be lowering and raising operator of KE in Eq.(1), respectively. For $cq < 0$ there are similar results. Thus the commutators of each member of this quartet with H, and with each other, do not generate any *new* operators and constitute a small Lie algebra within each sector. Once the sectors are individually diagonalized and their energies summed, the original model can be said to be solved in closed form.[1]

DISPERSION. In our expanded version of the model, the kinetic energy of a right-going particle takes the form $\varepsilon(k) = c\times(k-k_F)+h(k–k_F)$ subject only to some mild conditions on the *function* $h(k)$: that both it and its derivative $h'(k)$ vanish at $k=k_F$, to ensure that the Fermi level and the Fermi velocity are unaffected. (Similarly for left-goers, with $c\to -c$ and $k_F\to -k_F$.)[6] We note that once $h(k)$ ceases to be identically zero, no matter how large or small it might be relative to the original kinetic energy $c|k|$, the operators $a_\sigma(q)$ and $a^\dagger_{-\sigma}(-q)$ cease to be exact lowering or raising operators of KE, although H_2 is unaffected in its appearance. We expand the function h in a Taylor series about k_F :

$h(k-k_F)=h(0)+(k-k_F)h_1+\frac{1}{2!}(k-k_F)^2h_2+\frac{1}{3!}(k-k_F)^3h_3+\ldots$, requiring only that $h(0)=0$ and $h_1=0$ as well. A typical energy curve is shown in Fig. 1.

The existence of dispersion vitiates the algebra that solved the Luttinger model, but the one essential feature that does remain ultimately leads to a solution.

By conservation of momentum, the decomposition into sectors labeled q *remains completely valid*. We shall see that all results, including correlation functions, can be expanded in powers of q. Thus they must reduce to the corresponding Luttinger model formulas in the long wave-length “correspondence limit,” at $q\to 0$.

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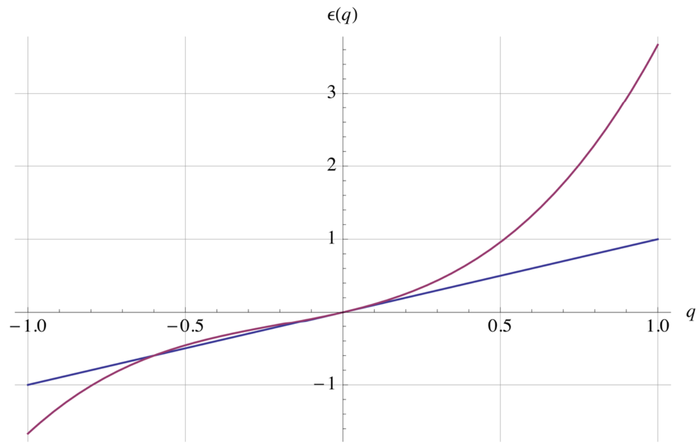

Figure 1. Kinetic energy $\varepsilon(q)$ of right-hand going fermions (curve) at small values of $q = k - k_F$ about the Fermi surface, compared to the original (straight-line dispersion) cq.

In this hypothetical example we have set $c = 1$, the second-derivative model parameter h_2=+2, the third-derivative model parameter h_3=+10, and all other h_j's = 0. Note the asymmetry in quasiparticle energy $|\varepsilon(q)|$ due to the mixing of even and odd terms. (The right-left symmetry is restored by the left-hand goers, but quasiparticle-quasihole symmetry about the Fermi level is broken.)

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NOTATION. In the expanded model, the old operators require a new notation:

$$a_{\sigma,0}(q)=\delta_{\frac{q}{|q|},\tau}\times\sqrt{\frac{2\pi}{L\,|\,q\,|}}\sum_{k=-\infty}^{+\infty}c^{\dagger}_{\sigma,\tau}(k-\tau\frac{q}{2})c_{\sigma,\tau}(k+\tau\frac{q}{2})\qquad. \tag{3A}$$

The above are the lowering operators. Their Hermitean conjugates are,

$$a^{\dagger}_{\sigma,0}(q)=\delta_{\frac{q}{|q|},\tau}\times\sqrt{\frac{2\pi}{L\,|\,q\,|}}\sum_{k=-\infty}^{+\infty}c^{\dagger}_{\sigma,\tau}(k+\tau\frac{q}{2})c_{\sigma,\tau}(k-\tau\frac{q}{2}) \tag{3B}$$

where $\tau=\pm 1$ refers to right- or left-goers. The operator $a_{\sigma,0}(q)$, formerly written $a_{\sigma}(q)$ in Eq. (2), is proportional to the density fluctuation operator of right-goers at momentum-transfer $q>0$ and of left-goers at momentum transfer $q<0$. (The new *subscript* “0” designates it as the first of a sequence of orthogonal fields numbered $j=0, 1, 2, \ldots$)

As first proved in ref.1 these boson operators satisfy $[a_{\sigma',0}(q'),\, a^{\dagger}_{\sigma,0}(q)\;]=\delta_{\sigma,\sigma'}\delta_{q,q'}$. With n or m standing for the composite label (σ, j), the fields in the *new* model also satisfy an extended commutator algebra, $[a_m(q),a^{\dagger}_n(q')]=\delta_{n,m}\delta_{q,q'}$ and $[a_m(q),a_n(q')]=0$. The proof that they do (in the “thermodynamic limit” $L\to\infty$) is given in Appendix A.

Except for the extended subscript the interaction Hamiltonian remains that of Eq.(2),

$$H_2=\frac{1}{2}\times\frac{1}{2\pi}\sum_{q=-\infty}^{+\infty}\sum_{\sigma}\sum_{\sigma'}V(q)\,|\,q\,|\Big(\big(a^{\dagger}_{\sigma,0}(q)+a_{-\sigma,0}(-q)\big)a_{\sigma',0}(q)+H.c.\Big)\qquad. \tag{4}$$

That is, the *appearance* of the interactions is not affected by the nonlinear dispersion introduced into the model problem.

EQUATIONS OF MOTION IN THE GENERALIZED MODEL. Once dispersion has been inserted into the kinetic energy, the corresponding kinetic energy Hamiltonian H_1 ,

$$H_1=\sum_{k}\sum_{\tau}\sum_{\sigma}\varepsilon_{\tau}(k)c^{\dagger}_{\sigma,\tau}(k)c_{\sigma,\tau}(k)\quad,\quad\text{where}\quad\varepsilon_{\tau}(k)=c\times(\tau k-k_F)+h(\tau k-k_F) \tag{5}$$

does change *its* appearance and its spectrum can no longer be represented by a single boson field. However, insofar as scattering processes from one branch to the other are excluded[5] H_1 can still be decomposed into a sum of right (+) and left (–) -goers' kinetic energy, as: $H_1 = \sum_{\tau=\pm 1} H_{1,\tau}$. Therefore we analyze just right-hand goers, $q > 0$, as the analysis is similar for left-hand goers, *mutatis mutandis*.

The equation of motion of $a_{\sigma,0}(q)$ introduces a new boson field $a_{\sigma,1}(q)$ sharing the same values of σ and q, as follows:

$$[a_{\sigma,0}(q), H_{1,+}] = [a_{\sigma,0}(q), H_1] = A_0^{(0)}(q)\, a_{\sigma,0}(q) + A_1^{(0)}(q)\, a_{\sigma,1}(q)\,. \tag{6}$$

The equation of motion of the new operator $a_{\sigma,1}(q)$ produces yet another, $a_{\sigma,2}(q)$,

$$[a_{\sigma,1}(q), H_{1,+}] = A_1^{(1)}(q)\, a_{\sigma,1}(q) + A_0^{(1)}(q)\, a_{\sigma,0}(q) + A_2^{(1)}(q)\, a_{\sigma,2}(q)\,, \tag{7}$$

and so on, *ad infinitum*. The coefficients $A_p^{(m)}$ in this iterative procedure can be written as an infinite-dimensional, symmetric, tridiagonal array:

$$\vec{\vec{\mathrm{A}}}(q) = \begin{pmatrix} A_0^{(0)} & A_1^{(0)} & 0 & 0 & 0 \ldots \\ A_1^{(0)} & A_1^{(1)} & A_2^{(1)} & 0 & \ldots \\ 0 & A_2^{(1)} & A_2^{(2)} & A_3^{(2)} \ldots & \\ 0 & 0 & A_3^{(2)} & \ldots & \end{pmatrix} . \tag{8}$$

At fixed q this matrix is isomorphic to that of the equations of motion of a string of masses and springs. To simplify notation we define a row-vector creation operator: $\vec{\alpha}_\sigma^\dagger(q) = (a_{\sigma,0}^\dagger(q), a_{\sigma,1}^\dagger(q), a_{\sigma,2}^\dagger(q), \ \ldots)$ and its conjugate column-vector, $\vec{\alpha}_\sigma(q) = (\vec{\alpha}_\sigma^\dagger(q))^\dagger$. This allows for a simplified notation, in which the motional energy takes the form, $H_{1,+} = \sum_{q>0} \sum_{\sigma} \vec{\alpha}_\sigma^\dagger(q) \cdot \vec{\vec{\mathrm{A}}}(q) \cdot \vec{\alpha}_\sigma(q)$. Similar manipulations at $q < 0$ take care of the left-goers. Finally we represent the kinetic energy H_1 as follows:

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$$H_1 = \sum_{q=-\infty}^{+\infty} \sum_{\sigma} \vec{\alpha}_{\sigma}^{\dagger}(q) \cdot \vec{\vec{\mathrm{A}}}(q) \cdot \vec{\alpha}_{\sigma}(q) \tag{9}$$

This H_1 is functionally equivalent to the fermion expression in Eq. (5).

It remains only to identify the individual members of the hierarchy and, simultaneously, to calculate the matrix elements in Eq. (8). Unlike high-energy string theory, here they are not given but have to be extracted from the equations of motion.

For the right-goers, upon substitution of sums by integrals, $\sum_k \Rightarrow \frac{L}{2\pi}\int dk$, the first diagonal entry in (8) is calculated as follows:

$$\begin{aligned} A_0^{(0)}(q) &= [a_{\sigma,0}^{\dagger}(q),[a_{\sigma,0}(q),H_1]] = cq + \frac{1}{q}\int_{k_F-|q|/2}^{k_F+|q|/2} dk\{h(k-k_F+q/2) - h(k-k_F-q/2)\} \\ &\approx cq + \frac{q^3}{12}\frac{d^3h(k-k_F)}{dk^3}\bigg|_{k_F} + O(q^5) \equiv cq + \frac{q^3}{12}h_3 + O(q^5) \end{aligned} \tag{10}$$

For left-goers, replace q by $-q$. Although $h_0 = h_1 = 0$ (by definition,) the coefficients h_2 , h_3, …, *are* significant and are, so far, arbitrary. Accordingly, from (6),

$$\begin{aligned} A_1^{(0)}(q)\,a_{\sigma,1}(q) &= [a_{\sigma,0}(q),H_1] - A_0^{(0)}(q)\,a_{\sigma,0}(q) = \\ &\sqrt{\frac{2\pi}{Lq}}\sum_k \{cq + h(k+q/2) - h(k-q/2) - A_0^{(0)}(q)\}\; c_{\sigma,+}^{\dagger}(k-\frac{q}{2})c_{\sigma,+}(k+\frac{q}{2}) \end{aligned} \tag{11}$$

Given that $a_{\sigma,1}(q)$ is normalized, it follows that

$$[A_1^{(0)}(q)\,a_{\sigma,1}(q), A_1^{(0)}(q)\,a^{\dagger}_{\sigma,1}(q)] = \left(A_1^{(0)}(q)\right)^2 =$$

$$\frac{2\pi}{Lq}\sum_k \{cq + h(k-k_F+q/2) - h(k-k_F-q/2) - A_0^{(0)}(q)\}^2$$

$$\times\{\tilde{n}_{\sigma,+}(k-k_F-\frac{q}{2}) - \tilde{n}_{\sigma,+}(k-k_F+\frac{q}{2})\ \}$$

$$= \frac{1}{q}\int_{k-q/2}^{k+q/2} dk \sum_k \{cq + h(k+q/2) - h(k-q/2) - A_0^{(0)}(q)\}^2$$

$$= \frac{1}{2}\int_{-1}^{1} dx\{(\frac{1}{2}q^2 h_2)x + (\frac{1}{12}q^3 h_3)(\frac{3}{2}x^2 - \frac{1}{2}) + O(q^4)\ldots\}^2$$

$$= \frac{(h_2)^2}{12}q^4 + \frac{(h_3)^2}{720}q^6 + \ldots \qquad (12)$$

as calculated[7] to leading orders in powers of q. Then,

$$A_1^{(1)}(q) = \left[[A_1^{(0)}(q)a_{\sigma,1}(q), H_1], A_1^{(0)}(q)a^{\dagger}_{\sigma,1}(q)\right] \ / \ \left(A_1^{(0)}(q)\right)^2$$

$$= cq + q^3(\frac{11}{60})h_3 + O(q^5) \qquad (13)$$

This Lanczös-type procedure is iterated until it is deemed to have converged and all significant entries to matrix (8) are made explicit.

EIGENVALUE EQUATION AT LONG WAVELENGTHS. If the interactions are not too strong and q is sufficiently small, the procedure of solving the equations of motion of the boson quadratic form $H = H_1 + H_2$ converges quickly. After a first iteration the boson frequencies are the two positive eigenvalues of the 4×4 matrix of coefficients, $\boldsymbol{M}$, exhibited in Table I. This matrix was calculated from the equations of motion at arbitrary h_2 and h_3 and small q. Its first 4 eigenvalues are $\pm\omega_0(q)$ and $\pm\omega_1(q)$. In second iteration (3-site strings) $\boldsymbol{M}$ has 3 branches and its 6 eigenvalues are $\pm\omega_0(q)$, $\pm\omega_1(q)$, $\pm\omega_2(q)$.

After N iterations $\boldsymbol{M}$ has dimension $2N$. If $q \to 0$, $N = 1$ suffices and Luttinger's model is recovered precisely.

The resulting frequencies are plotted in Fig. 2 in the case of three possible or plausible interaction potentials.

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TABLE I.

Matrix $\mathbf{M}$ *in first iteration*

$cq+\frac{q^3}{12}h_3+2V(q)\lvert q\rvert/\pi$	$2V(q)\lvert q\rvert/\pi$	$\sqrt{\frac{(h_2)^2}{12}q^4+\frac{(h_3)^2}{720}q^6}$	0
$-2V(q)\lvert q\rvert/\pi$	$-cq-\frac{q^3}{12}h_3-2V(q)\lvert q\rvert/\pi$	0	$-\sqrt{\frac{(h_2)^2}{12}q^4+\frac{(h_3)^2}{720}q^6}$
$\sqrt{\frac{(h_2)^2}{12}q^4+\frac{(h_3)^2}{720}q^6}$	0	$cq+q^3(\frac{11}{60})h_3$	0
0	$-\sqrt{\frac{(h_2)^2}{12}q^4+\frac{(h_3)^2}{720}q^6}$	0	$-cq-q^3(\frac{11}{60})h_3$

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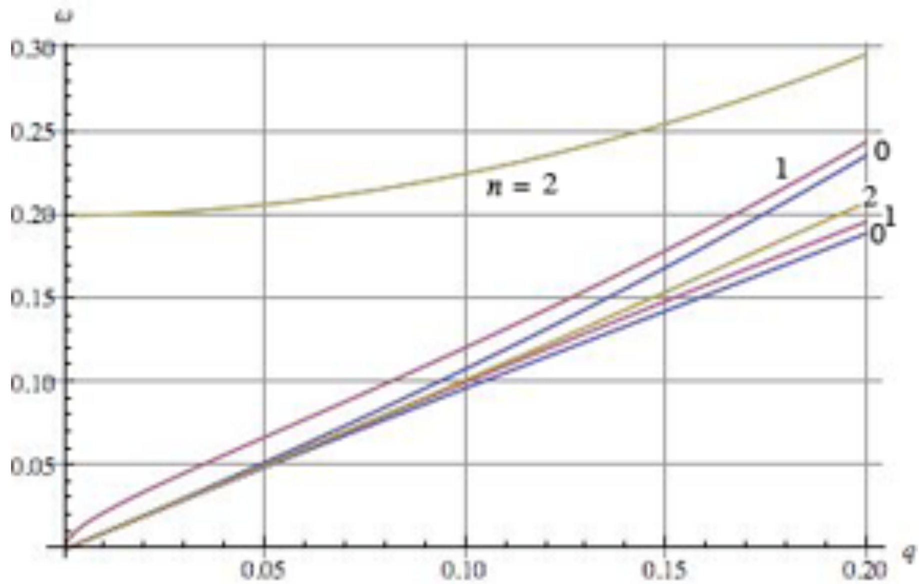

Figure 2. $\omega_0(q)$ and $\omega_1(q)$, eigenvalues of $\mathbf{M}$ of Table I, as functions of q, for two-fermion interactions of the types: $\frac{2V(q)}{\pi}|q| \equiv g|q|^{1-n}$, variously labeled n=0,1,2.

n = 0 corresponds to delta-function repulsion and n = 2 to Coulomb repulsion.

Here the (arbitrarily chosen) kinetic parameters are c = 1, h_2=2, h_3=10. As coupling constant we chose the relatively small value, g=0.02.

The lower three curves are representative of the infinite set of "string modes" that are relatively unaffected by the interactions, regardless of the value of g or n.

The upper three modes, consisting principally of density fluctuations, *are* strongly affected by interactions. *E.g.,* at a finite ω=0.20 observe the plasma frequency that appears already at q=0 for n=2, also the strong concavity of the upper curve labeled n=0.

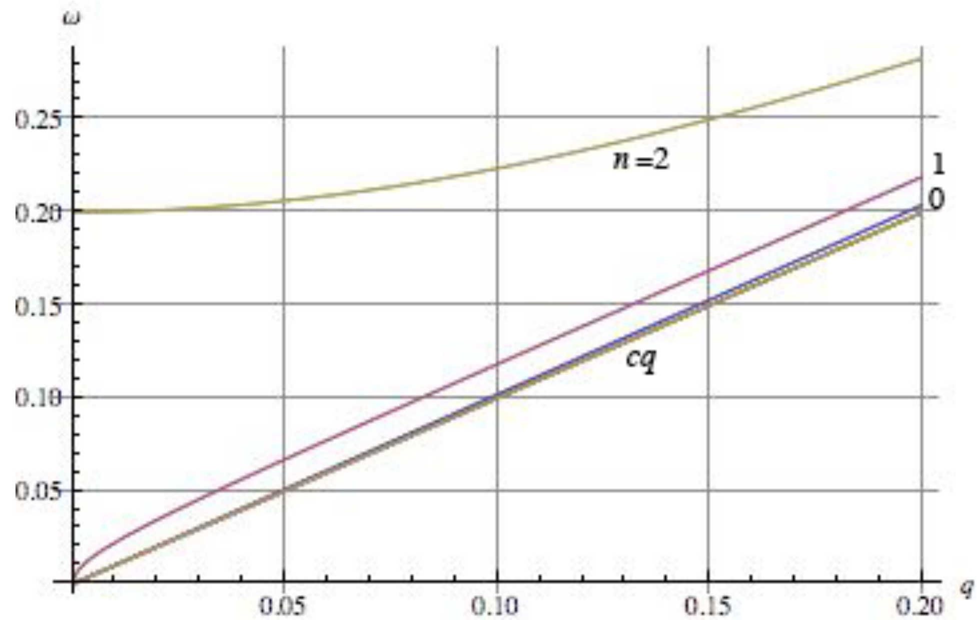

Figure 3. Luttinger-model solutions. Same as in Fig. 2 but with $h_2 = h_3 = 0$.

Individual curves are again labeled by their power-law interactions $n= 0, 1, 2$ (with the same coupling constant g=+0.02 as in Fig. 2).

Of the 3 upper curves, the straight line marked "n=0" (corresponding to repulsive delta function interactions) has slope *slightly* higher than c, as it should. The lower branches of Fig. 2 that do not appear in the original formulation[1] of the Luttinger model are here entirely decoupled from the interactions, and have each collapsed into one single line at exactly cq.

Note the upper branches are similar but not identical to those in Fig. 2, as they display marginally less upward curvature in the absence of dispersion.

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ATTRACTIVE *VS*. REPULSIVE FORCES. For inverse power laws $n = 1$ or 2, an attractive coupling constant ($g < 0$) leads to collapse of the system. One can still meaningfully examine the case of the delta function potential, $n = 0$, at either sign of g.

In the Luttinger model, an attractive delta function two-body interaction $-g\delta(x-x')$ lowers the speed of the collective normal mode below its original value c whereas a repulsive interaction raises it. This is the case in Fig. 3, although it is hard to see in this figure because of the small value of g. But regardless whether g is positive or negative, in the Luttinger model with delta-function interactions, all *renormalized speeds* are *independent* of q.

Let us rëexamine this feature in the expanded model with $c = 1$, $h_2=2$, $h_3=10$. Fig. 4 deals with both attractive and repulsive delta-function interactions (what we denoted $n = 0$ in Figs. 2 and 3) but in strong-coupling, using a more noticeable value of the coupling constant ($g=\pm 0.5$). We plot not the frequencies, but frequencies divided by the unperturbed cq, related to what one might term the relative "phase velocity."

At $g= -0.5$, maximal attraction, the phase velocity in the strongly affected collective mode drops markedly to near zero at $q=0$, and shows considerable dispersion at finite q. For $g=+0.5$, two-body repulsion, it rises (by 40%) because of the interaction, and exhibits some dispersion at finite q.

In both instances the relative phase velocity of the *second* solution hovers around 1, while displaying somewhat more upward curvature in the attractive than in the repulsive case.

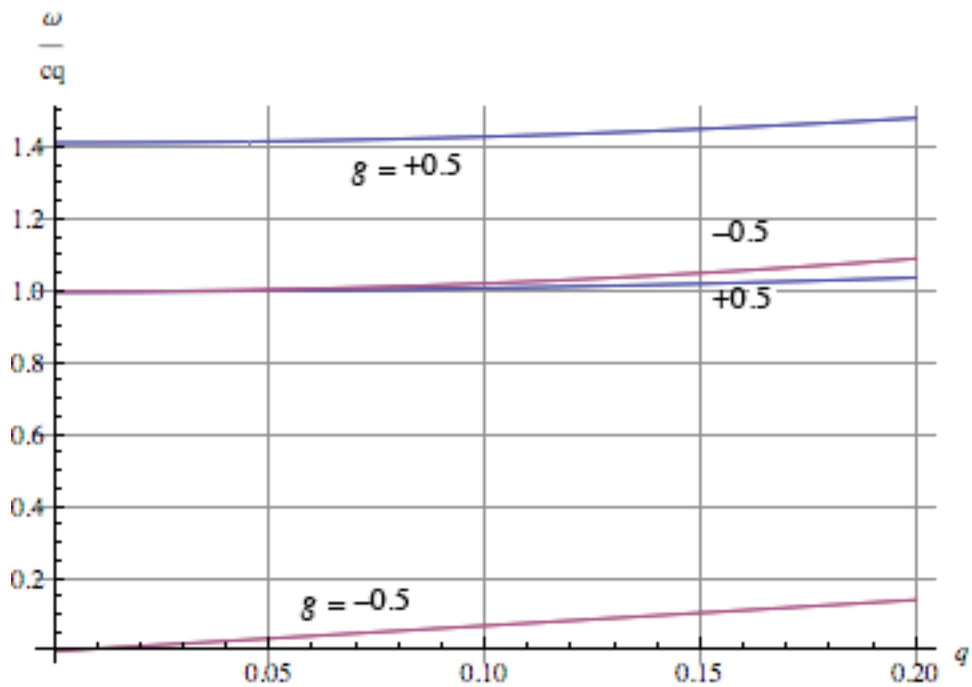

Figure 4. Relative Phase Velocities, $\omega(q)\div|cq|$ *vs.* q , calculated using Table I with h_2=2, h_3=10. (Delta-function repulsion or attraction (n=0) in strong coupling, assuming $g= \pm 0.5$.)

As $q \to 0$ the nontrivial modes (the one at $\omega(q)\div|cq| = 1.4$ for $g= +0.5$, the other at $\omega(q)\div|cq| = 0$ for $g= -0.5$) become identical to the limiting values found in the Luttinger model at the same values of the interactions.

The extra mode shown, one for each value of g, are absent in Luttinger's model. They have $\omega(q)\div|cq| = 1$ at $q \to 0$ but, at finite q, both show some extra upward dispersion reflecting the choice of h_2 and h_3. Higher modes, if we were to calculate them, would also lie close to $\omega(q)\div|cq|$ = 1 at small q.

CONCLUSION. The original Luttinger model[1] of interacting fermions in one dimension (and *only* that model) is, fortuitously, a quadratic form in a single field of boson operators, the a_0's.

Introducing nonlinear dispersion (*e.g.* mass) into the kinetic energy of fermions in the Luttinger model *hardly changes the results at long wavelengths* (for we have kept the speed at the Fermi surface the same as before,) but it does cause a novel set of orthonormal modes, a_1 , …, a_j … , to be injected into the problem. These extra modes are associated with an harmonic *string* at each q, a feature that enlarges and changes the Hilbert space qualitatively, not just quantitatively.

Such extra normal modes are (obviously) present in Tomonaga's original model[8] (although they have not been explicitly discussed before,) but they were excluded in Luttinger's *by construction*.

Some of the novel modes may be detectable experimentally, optically or by injection phenomena. To calculate them theoretically we study their equations of motion. In the present work we used a long-wavelength approximation to expand the properties of each string as powers of q. We expressed the results in terms of the curvature parameters h_2 and h_3 and powers of q. At small q a truncated two-site string was sufficient for present purposes. More generally, numerical procedures are required.

Once the frequencies $\omega_j(q)$ for j= 0, 1, 2, … are known, one easily constructs the Hamiltonian in a diagonal form:

$$H = \sum_{q=-\infty}^{+\infty} \sum_{\sigma} \vec{\alpha}_{\sigma}^{\dagger}(q) \cdot \vec{\vec{\mathrm{H}}}(q) \cdot \vec{\alpha}_{\sigma}(q) + \Delta E_0 \tag{11}$$

and calculates the renormalized vacuum energy ΔE_0. Here the Hamiltonian matrix is,

$$\vec{\vec{\mathrm{H}}}(q) = \begin{pmatrix} |\omega_0(q)| & 0 & 0 & 0 & \ldots \\ 0 & |\omega_1(q)| & 0 & 0 & \ldots \\ 0 & 0 & |\omega_2(q)| & 0 & \ldots \\ 0 & 0 & \ldots & & \end{pmatrix} . \tag{12}$$

In the present examples as illustrated in the figures, finite dispersion affected the normal modes only at finite or large q. That is a consequence of the restrictions put on the functions $h(k-k_F \pm q)$ near the Femi surface. For this reason in $\lim .|\omega_j(q)| \xrightarrow[q\to 0]{} cq$ the nontrivial solution $\omega_0(0)$ takes on a value which is the same as in the Luttinger model (*e.g.*, the plasma frequency ω_{pl} for n=2.) Similar decoupling would occur at short wavelengths, $q \to \infty$, if we enforced an extra requirement, $h(\pm\infty) \to 0$ (requiring that the dispersion becomes linear again asymptotically at $k \to \pm\infty$.)

Once we know the normal mode frequencies, the vacuum energy is, quite generally,

$$\Delta E_0 = \frac{1}{2} \sum_{all\ q} \sum_{j=0}^{\infty} (|\omega_j(q)| - c|q|) . \tag{13}$$

As for the inverse process, fermion operators $\Psi(x)$ are still defined in the same way as before;[2] however, the exponentiated bosons are no longer simple raising/lowering operators of H but, in the dispersive medium, are linear combinations of string operators.

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APPENDIX A.

Here we show that boson commutation relations are, at the very least, plausible for the various a's.[9]

Consider a Fermi sea of noninteracting right-going fermions, their Fermi level set at some k_F. An arbitrary bilinear form in fermion operators that decreases total momentum in an amount q, can be written in the form: $a(q) \equiv \sqrt{\frac{2\pi}{L|q|}} \sum_k \Phi(k,q) c^\dagger_{k-q/2} c_{k+q/2}$. If the a's are to be bosons, their amplitudes Φ need be normalized such that $[a(q), a^\dagger(q)] = 1$. That is, $\frac{1}{q}\int_{-q/2}^{+q/2} dk\,|\Phi(k,q)|^2 = \int_{-1/2}^{+1/2} dx\,|\Phi(xq,q)|^2 \equiv \langle|\Phi(k,q)|^2\rangle = 1$ is required, after the substitution of sums by integrals $\left(\sum_k \Rightarrow \frac{L}{2\pi}\int dk\right)$. An infinite number of acceptable linearly independent functions exist on the unit interval $(-\tfrac{1}{2} < x < \tfrac{1}{2})$ at every given q. To distinguish them let us index the quantities and operators by a subscript: $a_j(q) \equiv \sqrt{\frac{2\pi}{L|q|}} \sum_k \Phi_j(k,q) c^\dagger_{k-q/2} c_{k+q/2}$ and require these functions to form an orthonormal set on the unit interval: $\int_{-1/2}^{+1/2} dx\,\Phi^*_j(xq,q)\Phi_m(xq,q) = \delta_{j,m}$.

Operators constructed at a differing value of the momentum transfer, say at $q' \neq q$, are similarly designated $a_j(q')$. Generalizing the quantum statistics to arbitrary q we must also have $[a_n(q), a^\dagger_m(q')] = \delta_{n,m}\delta_{q,q'}$, as well as $[a_n(q), a_m(q')] = [a^\dagger_m(q'), a^\dagger_n(q)] = 0$ at all n, m, q, q'. Let's now prove what should be obvious.

$$[a_n(q),a_m(q')]=\frac{2\pi}{L\,|qq'|}\sum_k\{\Phi_n(k-q'/2,q)\Phi_m(k+q/2,q')$$
$$-\Phi_n(k+q'/2,q)\Phi_m(k-q/2,q')\}c^{\dagger}_{k-\frac{q+q'}{2}}c_{k+\frac{q+q'}{2}} \qquad \text{(A.1)}$$

The square of the normalization parameter $D_{n,m}$, is a dimensionless quantity:

$$D^2_{n,m}(q+q')\equiv$$
$$\frac{1}{|q+q'|}\int_{-\frac{q+q'}{2}}^{+\frac{q+q'}{2}}dk\,|\Phi_n(k-q'/2,q)\Phi_m(k+q/2,q')-\Phi_n(k+q'/2,q)\Phi_m(k-q/2,q')|^2$$

The *rhs* of Eq. (A.1) defines a new operator $b_{n,m}$. When normalized, such that $[b_{n,m}(q+q'),b^{\dagger}_{n,m}(q+q')]=1$, it is:

$$b_{n,m}(q+q')\equiv\frac{1}{|D_{n,m}(q+q')|}\sqrt{\frac{2\pi}{L\,|q+q'|}}\sum_k\{\Phi_n(k-q'/2,q)\Phi_m(k+q/2,q')$$
$$-\Phi_n(k+q'/2,q)\Phi_m(k-q/2,q')\}c^{\dagger}_{k-\frac{q+q'}{2}}c_{k+\frac{q+q'}{2}} \qquad \text{(A.2)}$$

Now, compare (A.1) and (A.2) to verify that the commutator in (A.1) is,

$$[a_n(q),a_m(q')]=\frac{1}{\sqrt{L}}\left(|D_{n,m}(q+q')|\sqrt{\frac{2\pi\,|q+q'|}{|qq'|}}\right)b_{n,m}(q+q') \quad . \qquad \text{(A.3)}$$

Let us test that the *rhs* of (A.3) actually vanishes. Consider,

$$\left[[a_n(q),a_m(q')],[a^{\dagger}_m(q'),a^{\dagger}_n(q)]\right]=\frac{1}{L}\left(D^2_{n,m}(q+q')\frac{2\pi\,|q+q'|}{|qq'|}\right) \qquad \text{(A.4)}$$

$D^2_{n,m}(q+q')$ is a finite, dimensionless number defined by the integral just below (A.1) and it is bounded. Hence the parenthetical expression in (A.4) is also bounded. Denote it: $l_{n,m}(q,q')$, a length independent of L. (Note: $l_{0,0}(q,q')=0$.) It follows that (A.3) and (A.4) vanish, the first as $\sqrt{l_{n,m}(q,q')/L}$ and the second as $l_{n,m}(q,q')/L$.

Various other commutation relations expected for continuous boson fields are all equally satisfied in the thermodynamic limit $L\to\infty$, as is shown by similar calculations.

FOOTNOTES AND REFERENCES

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[5] to keep the number of particles within each branch constant

[6] If it were additionally required that in each branch the energy $\varepsilon(k)$ remain single-valued, we could specify that $h'(k) > -c$ for the right-hand goers (and similarly, $h'(k) < c$ for left-goers) at *all* values of k, but this optional condition was not imposed in the small q expansion considered here.

[7] initial calculations are simplified somewhat upon observing that

$$\{cq + h(k+q/2) - h(k-q/2) - A_0^{(0)}(q)\}$$

$$= \{h(k-k_F+q/2) - h(k-k_F-q/2) - < h(k-k_F+q/2) - h(k-k_F-q/2) >\}$$, where

$$< f(k \pm q/2) > \equiv \frac{1}{q}\int_{-q/2}^{q/2} dk f(k \pm q/2)$$ for any function f.

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[9] for notational simplicity we leave out all unnecessary subscripts in this Appendix

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